AMS 229: Convex Optimization
Fall 2018

Midterm Exam

Name:______________________________ Student ID:______________________________

For this exam you only need a pen/pencil. No electronic device is allowed. Write your answers in the spaces provided. If you need more space, work on the other side of the page.

Problem 1 (10 + 5 + 5 = 20 points)

A real $n \times n$ matrix X is called doubly stochastic if its elements are non-negative, and all row sums and column sums equal unity, that is,

$$X_{ij} \geq 0, \qquad \sum_{i=1}^{n} X_{ij} = \sum_{j=1}^{n} X_{ij} = 1, \quad \text{for all } i, j = 1, \ldots, n.$$

In this problem, we investigate the set $\mathcal{X} = \{X \in \mathbb{R}^{n\times n} \mid X \text{ is doubly stochastic}\}$.

(a) Prove that $\mathcal{X}$ is a convex set.

Consider two matrices $X, Y \in \mathcal{X}$. Then for all $0 \leq \theta \leq 1$, the convex sum $\theta X + (1-\theta)Y$ satisfies

$$\theta X_{ij} + (1-\theta)Y_{ij} \geq 0, \quad \sum_{i=1}^{n} \theta X_{ij} + (1-\theta)Y_{ij} = \sum_{j=1}^{n} \theta X_{ij} + (1-\theta)Y_{ij} = 1,$$

meaning the matrix $\theta X + (1-\theta)Y \in \mathcal{X}$. Therefore, $\mathcal{X}$ is a convex set.

(b) Is $\mathcal{X}$ a cone? Why/why not?

No, it is not. For any $X \in \mathcal{X}$ and $\alpha > 0$, the scaled matrix αX satisfies $\alpha X_{ij} \geq 0$. However, $\sum_{i=1}^{n} \alpha X_{ij} = \sum_{j=1}^{n} \alpha X_{ij} = \alpha$, which is an arbitrary positive number. Thus, $\mathcal{X}$ is not a cone.

(c) Is $\mathcal{X}$ a polytope (also known as polyhedron)? Why/why not?

Yes, $\mathcal{X}$ is a polytope. This is because $\mathcal{X}$ is defined as the collection of linear inequalities and equalities. Geometrically, the convex polytope $\mathcal{X}$ is a subset of the $(n^2 - 2n + 1)$ dimensional affine supspace in $\mathbb{R}^{n^2}$, defined through the intersection of n^2 halfspaces and $2n$ hyperplanes.

Problem 2 (5 + 5 + 5 + 5 = 20 points)

Consider the following optimization problem, which we call problem (P). Here, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$ are given.

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \\ & x_i \in \{0,1\}, \quad i = 1,\dots,n. \end{aligned}$$

(a) Is problem (P) a convex optimization problem? Why/Why not?

No, Problem (P) is not convex. This is because the constraint set is non-convex due to the requirement that the elements of vector $\boldsymbol{x} \in \mathbb{R}^n$ be binary.

(b) Consider the following modification of problem (P), which we call problem (Q).

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \\ & 0 \le x_i \le 1, \quad i = 1,\dots,n. \end{aligned}$$

Is problem (Q) convex? If "yes", then what kind of convex optimization problem is it? If "no", then explain why.

Yes, problem (Q) is convex. This is because both the constraint set and the objective function are convex.
This is an LP since the objective is a linear function, and the constraint set consists of linear inequalities, i.e., an intersection of finite number of halfspaces.

(c) Suppose the optimal value for problem (P) is p_{opt}, and the same for problem (Q) is q_{opt}. Write an inequality between p_{opt} and q_{opt}. Justify your answer.

The inequality is $p_{\text{opt}} \ge q_{\text{opt}}$. This is because the constraint set of problem (P) is a proper subset of the constraint set of problem (Q), while both the problems share the same objective function.

(d) Suppose we do not know how to solve problem (P). Instead, we solve problem (Q), and suppose we find that its minimizer $\boldsymbol{x}^q_{\text{opt}} \in \{0,1\}^n$ (vector of size $n \times 1$ whose each element is binary). Can we then conclude that $p_{\text{opt}} = q_{\text{opt}} = \boldsymbol{c}^\top \boldsymbol{x}^q_{\text{opt}}$, that is, problem (Q) solves problem (P)? Explain.

Yes, because problem (Q) subsumes problem (P), as explained in part (c).

Problem 3 (5 $\times$ 2 = 10 points)

For each the following statements, ONLY ONE among the three options are correct. Choose the correct option for each. You DO NOT need to provide any explanation.

(a) For a non-convex function f, its tri-conjugate f^{***} satisfies
(i) $f^{***} = f$, the original non-convex function
(ii) $f^{***} = f^{*}$, which is a convex function
(iii) $f^{***} = f^{**}$, which is a convex function

(ii) since $f^{***} = (f^{**})^{*} = (f^{*})^{**} = f^{*}$, which is convex.

(b) At each point on the boundary of a compact convex set, a supporting hyperplane
(i) may not exist
(ii) exists
(iii) exists and is unique

(ii) due to supporting hyperplane theorem. Uniqueness may fail (e.g., at the vertices of polytopes).

(c) The negative entropy function $f(\boldsymbol{x})$ given by

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i \log x_i, \qquad \mathrm{dom}(f) = \{\boldsymbol{x} \in \mathbb{R}^n_{>0} \mid \mathbf{1}^\top \boldsymbol{x} = 1\},$$

(i) is quasiconcave
(ii) is quasiconvex but not convex
(iii) both quasiconvex and convex

(iii) since it is a convex function (can be checked by Hessian, for example) and therefore also quasiconvex.

(d) A function is concave if and only if its hypograph is
(i) convex
(ii) open
(iii) closed

(i). This is graphically obvious.

(e) The dual cone of $\mathbb{S}^n_+$ equals
(i) $\mathbb{S}^n_+$
(ii) $\mathbb{S}^n_-$
(iii) $\mathbb{S}^n$

(i) since the cone is self-dual. See lecture 5, p. 5.

Some useful information

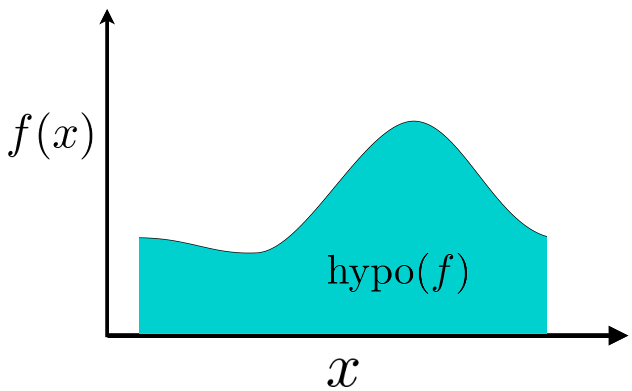
- Convex conjugate or Legendre-Fenchel transform of function $f(\boldsymbol{x})$ is

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x} \in \operatorname{dom}(f)} \left(\boldsymbol{y}^\top \boldsymbol{x} - f(\boldsymbol{x}) \right).$$

  f^* is convex even if f is not.

- Suppose f is a <u>twice differentiable</u> function. Then f is convex (resp. concave) if and only if $\operatorname{dom}(f)$ is convex, and Hessian of f is positive (resp. negative) semidefinite everywhere in $\operatorname{dom}(f)$.

- Hypograph of a function f is the set of points lying on or below its graph.



- Given a cone $\mathcal{K}$, its dual cone $\mathcal{K}^*$ is given by

$$\mathcal{K}^* = \{y \mid \langle y, x \rangle \geq 0, \text{ for all } x \in \mathcal{K}\}.$$

  $\mathcal{K}^*$ is convex even if $\mathcal{K}$ is not.